A REMARK ON LITTLEWOOD-PALEY PROJECTIONS

YOUNGHUN HONG

Abstract. We establish the kernel estimates for the Littlewood-Paley projections associated with a Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^3$ for a large class of short-range potentials $V(x)$. As a corollary, we prove the homogeneous Sobolev inequality.

1. Introduction

1.1. **Littlewood-Paley projections.** Consider a Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^3$. We define the potential class $\mathcal{K}_0$ as the norm closure of bounded, compactly supported functions with respect to the global Kato norm

$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} dy.$$

A complex number $\lambda \in \mathbb{C}$ is called a *resonance* if the equation $\psi + (-\Delta - \lambda \pm i0)^{-1} V\psi = 0$ has a slowly decaying solution such that $\psi \notin L^2$ but $\langle x\rangle^{-s}\psi \in L^2$ for all $s > \frac{1}{2}$. Throughout the paper, we assume that $V \in \mathcal{K}_0$ and H has no eigenvalues or resonances on the positive real line $[0, +\infty)$. The potential class $\mathcal{K}_0$ is known as a critical potential class for the dispersive estimate for the linear propagator e^{itH} [3, 10].

Under the above assumptions, it is known that H is self-adjoint on L^2 and that its spectrum $\sigma(H)$ is purely absolutely continuous on the positive real-line $[0, +\infty)$ and has at most finitely many negative eigenvalues [3]. Moreover, for a bounded Borel function $m : \sigma(H) \to \mathbb{C}$, one can define an L^2-bounded operator $m(H)$ via the functional calculus. Note that if $V = 0$, such a multipliers is simply a Fourier multiplier $(m(-\Delta)f)^\wedge(\xi) = m(|\xi|^2)\hat{f}(\xi)$.

In this paper, we investigate the kernel estimates for the Littlewood-Paley projections associated with a Schrödinger operator H. Let $\chi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \subset [\frac{1}{2}, 2]$ and $\sum_{N\in 2^{\mathbb{Z}}} \chi_N \equiv 1$ on $(0, +\infty)$, where $\chi_N := \chi(\frac{\cdot}{N})$. Let $\tilde{\chi}_N \in C_c^\infty(\mathbb{C})$ such that $\tilde{\chi}_N = \chi_N(\sqrt{\lambda})$ for $\lambda \in (0, +\infty)$ and $\tilde{\chi}_N(\lambda_j) = 0$ for all negative eigenvalues λ_j. We define a family of *Littlewood-Paley projections associated with* H by

$$\mathcal{P}_N := \tilde{\chi}_N(H).$$

When $V = 0$, we denote $P_N = \tilde{\chi}_N(-\Delta)$, which is the standard Littlewood-Paley projection.

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1.2. **Statement of the main results.** The first main theorem of this paper says that if V is small, $\mathcal{P}_N$ satisfies exactly the same kernel estimate as the homogeneous case $V=0$. For an integral operator T, we denote its kernel by $T(x,y)$.

Theorem 1.1 (Kernel estimate for $\mathcal{P}_N$: small potential)**.** *Suppose that $V\in\mathcal{K}_0$ and H has no eigenvalues or resonances on $[0,+\infty)$ and $\|V\|_{\mathcal{K}}<4\pi$. Then, for N and $m\geq 0$,*

$$|\mathcal{P}_N(x,y)|\lesssim\frac{N^3\|\chi\|_{W^{m,1}}}{\langle N(x-y)\rangle^{m+1}},$$

where $\|f\|_{W^{m,1}}:=\sum_{k=0}^m\|\nabla^k f\|_{L^1(\mathbb{R})}$.

Second, without the smallness assumption, we prove the following kernel estimates:

Theorem 1.2 (Kernel estimate for $\mathcal{P}_N$)**.** *Suppose that $V\in\mathcal{K}_0$ and H has no eigenvalues or resonances on $[0,+\infty)$.*
(i) (High frequencies) There exists $N_1=N_1(V)\gg 1$ such that for $N\geq N_1$ and $m\geq 0$,

$$|\mathcal{P}_N(x,y)|\lesssim\frac{N^3\|\chi\|_{W^{2m,1}}^{1/2}}{\langle N(x-y)\rangle^{\frac{2m+1}{2}}}.$$

(ii) (Low frequencies) There exist $N_0=N_0(V)\ll 1$ and $K(x_1,y)\in L_y^\infty L_{x_1}^1$ such that for $N\leq N_0$ and $m\geq 0$,

$$|\mathcal{P}_N(x,y)-P_N(x,y)|\lesssim\int_{\mathbb{R}^3}\frac{N^2\|\chi\|_{W^{2m,1}}^{1/2}K(x_1,y)}{|x-x_1|\langle N(x-x_1)\rangle^m}dx_1,$$

(iii) (Medium frequencies) For $N_0<N<N_1$, there exists $K_N(x_1,y)\in L_y^\infty L_{x_1}^1$ such that for $m\geq 0$,

$$|\mathcal{P}_N(x,y)-P_N(x,y)|\lesssim\int_{\mathbb{R}^3}\frac{N^2\|\chi\|_{W^{2m,1}}^{1/2}K_N(x_1,y)}{|x-x_1|\langle N(x-x_1)\rangle^m}dx_1,\ \sup_N\|K_N(x_1,y)\|_{L_y^\infty L_{x_1}^1}<\infty.$$

As a corollary, we have:

Corollary 1.3 (Boundedness for $\mathcal{P}_N$)**.** *If $V\in\mathcal{K}_0$ and H has no eigenvalues or resonances on $[0,+\infty)$, then*

$$\|\mathcal{P}_N f\|_{L^q}\lesssim N^s\|f\|_{L^p},\tag{1.1}$$

where $1\leq p\leq q\leq\infty$, $0\leq s<3$ and $\frac{1}{q}=\frac{1}{p}-\frac{s}{3}$.

Remark 1.4. (i) D'Ancona-Pierfelice proved the L^p-boundedness of $\mathcal{P}_N$ based on the Gaussian heat kernel estimate for the semigroup e^{-tH}. However, it required smallness of the negative part of the potential, $V_-(x):=\min(V(x),0)$ [6, Proposition 5.2].
(ii) The kernel estimates in Theorem 1.1 and 1.2 are stronger than the L^p-L^q bound (1.1). Indeed, Corollary 1.3 is not sufficient to prove the Sobolev inequality (see Theorem 1.6 below).

We prove Theorem 1.1 and 1.2 by the following strategy. First, we write the Littlewood-Paley projection as a formal series expansion

$$\mathcal{P}_N \text{``}=\text{''} \sum_{n=0}^{\infty} \mathcal{P}_N^n,$$

whose term $\mathcal{P}_N^n$ has explicit integral representation. We then prove that each $\mathcal{P}_N^n(x,y)$ satisfies the kernel estimate of the form in Theorem 1.2 and 1.3, respectively (Lemma 2.2, Lemma 4.4 and Lemma 5.1). Next, we prove that $\mathcal{P}_N^n(x,y)$ is summable in n (Lemma 3.1, Lemma 4.5, and Lemma 5.2). Finally, combining them, we complete the proof.

In this paper, we adopt the techniques to show dispersive estimates for a linear propagator e^{itH}. This approach allows us to take a lot of advantages from its recent progress. The first dispersive estimate of the form $\|e^{itH}P_c\|_{L^1\to L^\infty} \lesssim |t|^{-3/2}$ was first proved by Journé-Soffer-Sogge [11] under suitable assumptions on potentials. These restrictions have been relaxed by Rodnianski-Schlag [12], Goldberg-Schlag [8] and Goldberg [7]. Recently, employing the operator-valued Wiener theorem, Beceanu-Goldberg established the dispersive estimate for the scaling-critical potential class $\mathcal{K}_0$ [3, 10].

1.3. **Homogeneous Sobolev inequality.** The "inhomogeneous" Sobolev inequality associated with H is known for a fairly general class of potentials. We say that a potential V is *in Kato class* if

$$\lim_{r\to 0+} \sup_{x\in\mathbb{R}^3} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|} dy = 0,$$

and V is *locally in Kato class* if $V1_{|\cdot|\le R}$ is in Kato class for all $R>0$.

Theorem 1.5 (Inhomogeneous Sobolev inequality [13, Theorem B.2.1])**.** *Decompose $V = V_+ - V_-$ with $V_+, V_- \ge 0$. Suppose that V_- is in Kato class and V_+ is locally in Kato class. Then, for $\operatorname{Re} z < \inf\sigma(H)$,*

$$\|(H-z)^{-\frac{s}{2}} f\|_{L^q} \lesssim \|f\|_{L^p} \tag{1.2}$$

where $1<p<q<\infty$, $0<s<3$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{3}$.

When $V=0$, one can take $z=0$ in (1.3), called the "homogeneous" Sobolev inequality, or the fractional integration inequality. The proof of Theorem 1.5 is based on the heat kernel estimate, which requires $\operatorname{Re} z$ to be strictly less than the bottom of $\sigma(H)$ [13, Theorem B.7.1 and B.7.2].

As an application of Theorem 1.3, we prove the following homogeneous Sobolev inequality ($z=0$) without using the heat kernel estimate:

Theorem 1.6 (Homogeneous Sobolev inequality)**.** *If $V\in\mathcal{K}_0$ and H has no eigenvalues or resonances on $[0,+\infty)$, then*

$$\|H^{-\frac{s}{2}} P_c f\|_{L^q} \lesssim \|f\|_{L^p}, \tag{1.3}$$

where $1<p<q<\infty$, $0<s<3$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{3}$.

Remark 1.7. Yajima [15] proved that the wave operator $W := \underset{t\to+\infty}{s\text{-}\lim} e^{itH}e^{-it(-\Delta)}$ is bounded on $W^{2,p}$, when zero is not an eigenvalue or a resonance and $|\nabla^k V(x)| \lesssim \langle x\rangle^{-(5+)}$ for all multi-index k with $|k| \le 2$. Recently, Beceanu [2] extended this result to the scaling-critical class of potentials

$$B := \{V : \sum_{N\in 2^{\mathbb{Z}}} N^{1/2}\|V(x)\|_{L_x^2(N\le|x|\le 2N)} < \infty\}.$$

The homogeneous Sobolev inequality then follows from the boundedness and the intertwining property of the wave operators. Theorem 1.6 improves this consequence in that $\mathcal{K}_0$ is a larger class than B .

1.4. **Notations.** We denote the formal identity by $A\text{``}=\text{''}B$ which will be proved later.

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2. Proof of Theorem 1.1: Small Potential

For $z \notin \sigma(H)$, we define the resolvent by $R_V(z) := (H-z)^{-1}$, and denote $R_V^{\pm}(\lambda) := \lim_{\epsilon\to 0+} R_V(\lambda \pm i\epsilon)$. Then, by the Stone's formula, $\mathcal{P}_N$ can be written as

$$\mathcal{P}_N = \frac{1}{\pi}\int_0^\infty \chi_N(\sqrt{\lambda})\,\mathrm{Im}\,R_V^+(\lambda)d\lambda. \tag{2.1}$$

Iterating the resolvent identities, we get a formal Born series of the resolvent operator:

$$R_V^+(\lambda) = R_0^+(\lambda)(I+VR_0^+(\lambda))^{-1}\text{``}=\text{''}\sum_{n=0}^\infty (-1)^n R_0^+(\lambda)(VR_0^+(\lambda))^n.$$

Plugging this formal series into the spectral representation (2.1) and by a change of variables $\lambda \mapsto \lambda^2$, we obtain a formal series expansion of $\mathcal{P}_N$:

$$\begin{aligned}\mathcal{P}_N &= \frac{1}{\pi}\int_0^\infty \chi_N(\sqrt{\lambda})\,\mathrm{Im}[R_V^+(\lambda)]d\lambda\\ &\text{``}=\text{''}\sum_{n=0}^\infty(-1)^n\frac{N}{\pi}\int_{\mathbb{R}}\varphi_N(\lambda)\,\mathrm{Im}[R_0^+(\lambda^2)(VR_0^+(\lambda^2))^n]d\lambda =: \sum_{n=0}^\infty \mathcal{P}_N^n,\end{aligned} \tag{2.2}$$

where $\varphi(\lambda)$ is the odd extension of $\lambda\chi 1_{[0,+\infty)}$ to $\mathbb{R}$ and $\varphi_N = \varphi(\frac{\cdot}{N})$. The main advantage of this formal series expansion is that one can express the kernel of each $\mathcal{P}_N^n$ explicitly in the integral form by the free resolvent formula $R_0^+(\lambda)(x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$.

Lemma 2.1 (Integral representation of $\mathcal{P}_N^n(x,y)$)**.**

$$\mathcal{P}_N^n(x,y) = (-1)^n\frac{N}{\pi i}\int_{\mathbb{R}^{3n}}\frac{\prod_{k=1}^n V(x_k)}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}\check{\varphi}_N(\sigma_n)d\mathbf{x}_{(1,n)} \tag{2.3}$$

where $x_0 := x$, $x_{n+1} := y$, $d\mathbf{x}_{(1,n)} := dx_1\cdots dx_n$ *and* $\sigma_n := \sum_{j=0}^n |x_j - x_{j+1}|$.

Proof. Observe that $\mathcal{P}_N^n$ involves $(n+1)$ free resolvent operators, each of which is a convolution operator. Hence, by the free resolvent formula and the Fubini theorem, we write

$$\begin{aligned}
\mathcal{P}_N^n(x,y) &= (-1)^n\frac{N}{\pi}\int_{\mathbb{R}}\mathrm{Im}\int_{\mathbb{R}^{3n}}\varphi_N(\lambda)\frac{\prod_{k=0}^n e^{i\lambda|x_k-x_{k+1}|}}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}\prod_{k=1}^n V(x_k)d\mathbf{x}_{(1,n)}d\lambda\\
&= (-1)^n\frac{N}{\pi}\int_{\mathbb{R}^{3n}}\frac{\prod_{k=1}^n V(x_k)}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}\Big\{\mathrm{Im}\int_{\mathbb{R}}\varphi_N(\lambda)e^{i\lambda\sum_{j=0}^n|x_j-x_{j+1}|}d\lambda\Big\}d\mathbf{x}_{(1,n)}\\
&= (-1)^n\frac{N}{\pi i}\int_{\mathbb{R}^{3n}}\frac{\prod_{k=1}^n V(x_k)}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}\Big\{\int_{\mathbb{R}}\varphi_N(\lambda)e^{i\lambda\sigma_n}d\lambda\Big\}d\mathbf{x}_{(1,n)}\\
&= (-1)^n\frac{N}{\pi i}\int_{\mathbb{R}^{3n}}\frac{\prod_{k=1}^n V(x_k)}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}\check{\varphi}_N(\sigma_n)d\mathbf{x}_{(1,n)}.
\end{aligned}$$

In the third equality, we used the fact φ_N is an odd function. □

Theorem 1.1 is an immediate consequence of Lemma 2.2:

Lemma 2.2 (Off-diagonal decay estimate)**.**

$$|\mathcal{P}_N^n(x,y)|\lesssim(n+1)\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^n\frac{N^3\|\chi\|_{W^{m,1}}}{\langle N(x-y)\rangle^{m+1}}. \tag{2.4}$$

Proof of Theorem 1.1, assuming Lemma 2.2. Since $\|V\|_{\mathcal{K}}<4\pi$, the right hand side of (2.4) is summable in n. The formal series (2.2) is absolutely convergent and the sum satisfies the kernel estimate. □

Proof of Lemma 2.2. We claim that, in (2.3),

$$|\check{\varphi}_N(\sigma_n)|\lesssim\frac{N^2\sigma_n\|\chi\|_{W^{m,1}}}{\langle N(x-y)\rangle^{m+1}}. \tag{2.5}$$

Since φ is odd, by the trivial inequality $|\sin t|\le|t|$, we get

$$|\check{\varphi}_N(\sigma_n)|=\Big|\int_{\mathbb{R}}\varphi_N(\lambda)\sin(\lambda\sigma_n)d\lambda\Big|\le\sigma_n\int_{\mathbb{R}}\lambda\varphi_N(\lambda)d\lambda=\frac{N^2\sigma_n}{2}\int_{\mathbb{R}}\lambda^2\chi(\lambda)d\lambda\lesssim N^2\sigma_n\|\chi\|_{L^1}.$$

On the other hand, since $\varphi_N=\varphi(\frac{\cdot}{N})$, $\check{\varphi}\in\mathcal{S}$ and $\varphi(\lambda)=\lambda\chi(\lambda)$, we have

$$|\check{\varphi}_N(\sigma_n)|=N|\check{\varphi}(N\sigma_n)|\lesssim\frac{N}{(N\sigma_n)^m}|(\nabla^m\varphi)^\vee(N\sigma_n)|\le\frac{N^2\sigma_n\|\nabla^m\varphi\|_{L^1}}{(N\sigma_n)^{m+1}}\lesssim\frac{N^2\sigma_n\|\chi\|_{W^{m,1}}}{(N|x-y|)^{m+1}},$$

where the last inequality follows from the triangle inequality $\sigma_n\ge|x-y|$. Combining these two estimates, we prove the claim. Now we apply (2.4) to (2.3), then

$$|\mathcal{P}_N^n(x,y)|\lesssim\frac{N^3\|\chi\|_{W^{m,1}}}{\langle N(x-y)\rangle^{m+1}}\Big(\sum_{j=0}^n\int_{\mathbb{R}^{3n}}\frac{\prod_{k=1}^n|V(x_k)|}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}|x_j-x_{j+1}|d\mathbf{x}_{(1,n)}\Big).$$

It suffices to show that for each $0\le j\le n$,

$$\int_{\mathbb{R}^{3n}}\frac{\prod_{k=1}^n|V(x_k)|}{\prod_{k=0}^n 4\pi|x_k-x_{k+1}|}|x_j-x_{j+1}|d\mathbf{x}_{(1,n)}\lesssim\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^n.$$

To see this, we write the integral as

$$\Big(\int_{\mathbb{R}^{3j}}\frac{\prod_{k=1}^{j}|V(x_k)|}{\prod_{k=0}^{j-1}4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(1,j)}\Big)\Big(\int_{\mathbb{R}^{3(n-j)}}\frac{\prod_{k=j+1}^{n}|V(x_k)|}{\prod_{k=j+1}^{n}4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(j+1,n)}\Big).$$

By the definition of the global Kato norm and the Hölder inequalities, we bound the first term by

$$\begin{aligned}&\Big(\sup_{x_0\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(x_1)|}{4\pi|x_0-x_1|}dx_1\Big)\Big(\sup_{x_1\in\mathbb{R}^3}\int_{\mathbb{R}^{3(j-1)}}\frac{\prod_{k=2}^{j}|V(x_k)|}{\prod_{k=1}^{j-1}4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(2,j)}\Big)\\&\le\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)\Big(\sup_{x_1\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(x_2)|}{4\pi|x_1-x_2|}dx_2\Big)\Big(\sup_{x_2\in\mathbb{R}^3}\int_{\mathbb{R}^{3(j-2)}}\frac{\prod_{k=3}^{j}|V(x_k)|}{\prod_{k=2}^{j-1}4\pi|x_k-x_{k+1}|}d\mathbf{x}_{(3,j)}\Big)\\&\le\cdots\le\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^j,\end{aligned}$$

and similarly we bound the second term by $\big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\big)^{n-j}$. □

3. Proof of Theorem 1.2 (i): High Frequencies

Consider a large potential. Note that if $\|V\|_{\mathcal{K}}\ge 4\pi$, the sum of the right hand side of (2.4) is not convergent anymore. However, if N is large enough, one can still make the formal series (2.2) absolutely convergent by the following lemma:

Lemma 3.1 (Summability of $\mathcal{P}_N^n(x,y)$)**.** *For $\epsilon>0$, there exists a large number $N_1\gg 1$ such that for $N\ge N_1$,*

$$|\mathcal{P}_N^n(x,y)|\lesssim_{\|V\|_{\mathcal{K}},\epsilon}(n+1)\epsilon^nN^3. \tag{3.1}$$

Proof of Theorem 1.2 (i), assuming Lemma 3.1. For $\epsilon=\|V\|_{\mathcal{K}}^{-1}$, choose $N_1\gg 1$ from Lemma 3.1. For $N\ge N_1$, combining (2.4) and (3.1), we get

$$\begin{aligned}|\mathcal{P}_N^n(x,y)|&\lesssim\Big((n+1)\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^n\frac{N^3\|\chi\|_{W^{2m,1}}}{\langle N(x-y)\rangle^{2m+1}}\Big)^{1/2}\Big((n+1)\epsilon^nN^3\Big)^{1/2}\\&=\frac{n+1}{(4\pi)^{n/2}}\frac{N^3\|\chi\|_{W^{2m,1}}^{1/2}}{\langle N(x-y)\rangle^{\frac{2m+1}{2}}}.\end{aligned}$$

Summing them up, we prove Theorem 1.2 (i). □

To show Lemma 3.1, we recall the free resolvent estimate.

Lemma 3.2 (Free resolvent estimate [9, Lemma 2.1])**.**

$$\|R_0^+(\lambda^2)\|_{L^{4/3}\to L^4}\lesssim\langle\lambda\rangle^{-1/2}.$$

From this lemma, one can make $(VR_0^+(\lambda^2))^2$ arbitrarily small in $\mathcal{L}(L^1)$ for $\lambda\gg 1$:

Lemma 3.3. *Let $V \in \mathcal{K}_0$. Then,*

$$\sup_{\lambda\in\mathbb{R}} \|VR_0^+(\lambda^2)\|_{\mathcal{L}(L^1)} \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}, \tag{3.2}$$

$$\lim_{\lambda\to\infty} \|(VR_0^+(\lambda^2))^2\|_{\mathcal{L}(L^1)} = 0. \tag{3.3}$$

Proof. By the Minkowski inequality,

$$\|VR_0^+(\lambda^2)f\|_{L^1} \leq \int_{\mathbb{R}^3} \Big\|\frac{V(x)}{4\pi|x-y|}\Big\|_{L^1_x} |f(y)|dy \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}\|f\|_{L^1}.$$

Given $\epsilon > 0$, choose a bounded and compactly supported V_ϵ such that $\|V - V_\epsilon\|_{\mathcal{K}} < \epsilon$. As above, we approximate $(VR_0^+(\lambda^2))^2$ by $(V_\epsilon R_0^+(\lambda^2))^2$ with $O(\epsilon)$-error in $\mathcal{L}(L^1)$. For $(V_\epsilon R_0^+(\lambda^2))^2$, applying Lemma 3.2 and the Hölder inequalities, we get

$$\begin{aligned}
&\|(V_\epsilon R_0^+(\lambda^2))^2 f\|_{L^1} \lesssim \|V_\epsilon\|_{L^{4/3}}\|R_0^+(\lambda^2)V_\epsilon R_0^+(\lambda^2)f\|_{L^4} \lesssim \langle\lambda\rangle^{-1/2}\|V_\epsilon R_0^+(\lambda^2)f\|_{L^{4/3}}\\
&\lesssim \langle\lambda\rangle^{-1/2}\|V_\epsilon\|_{L^\infty}\|R_0^+(\lambda^2)f\|_{L^{4/3}_{x\in\operatorname{supp}V_\epsilon}} \lesssim \langle\lambda\rangle^{-1/2}\int_{\mathbb{R}^3}\Big\|\frac{1}{|x-y|}\Big\|_{L^{4/3}_{x\in\operatorname{supp}V_\epsilon}}|f(y)|dy\\
&\lesssim \langle\lambda\rangle^{-1/2}\|f\|_{L^1} \to 0 \text{ as } \lambda\to\infty.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain (3.3). □

Now we are ready to prove Lemma 3.1:

Proof of Lemma 3.1. Fix $\epsilon > 0$. It suffices to show that for $N \geq N_1$,

$$\|\mathcal{P}_N^n f\|_{L^\infty} \lesssim (n+1)\epsilon^n N^3\|f\|_{L^1} \tag{3.4}$$

First, by (3.3) and duality, we take $N_1 = N_1(\epsilon, V) \gg 1$ such that if $N \geq N_1$,

$$\|(VR_0^+(\lambda^2))^2\|_{\mathcal{L}(L^1)} \leq \epsilon^2;\ \|(R_0^-(\lambda^2)V)^2\|_{\mathcal{L}(L^\infty)} \leq \epsilon^2 \tag{3.5}$$

for all $\lambda \in \operatorname{supp}\varphi_N$. We split

$$\operatorname{Im}[R_0^+(\lambda^2)(VR_0^+(\lambda^2))^n] = R_0^+(\lambda^2)(VR_0^+(\lambda^2))^n - R_0^-(\lambda^2)(VR_0^-(\lambda^2))^n$$

in (2.1) into $(n+1)$ terms

$$\begin{aligned}
&(R_0^+(\lambda^2) - R_0^-(\lambda^2))(VR_0^+(\lambda^2))^n + R_0^-(\lambda^2)V(R_0^+(\lambda^2) - R_0^-(\lambda^2))(VR_0^+(\lambda^2))^{n-1}\\
&+\cdots+(R_0^-(\lambda^2)V)^n(R_0^+(\lambda^2) - R_0^-(\lambda^2)).
\end{aligned} \tag{3.6}$$

If $\lambda \in \operatorname{supp}\varphi_N$, by the mean value theorem, we have

$$\|(R_0^+(\lambda^2) - R_0^-(\lambda^2))\|_{L^1\to L^\infty} \leq \Big\|\frac{e^{i\lambda|x-y|} - e^{-i\lambda|x-y|}}{4\pi|x-y|}\Big\|_{L^\infty_y L^\infty_x} \leq \frac{|\lambda|}{2\pi} \lesssim N. \tag{3.7}$$

Thus, applying (3.2), (3.5) and (3.7) to each term in (3.6), we obtain

$$\begin{aligned}
&\|(R_0^-(\lambda^2)V)^k(R_0^+(\lambda^2) - R_0^-(\lambda^2))(VR_0^+(\lambda^2))^{n-k}f\|_{L^\infty}\\
&\lesssim \epsilon^{2\lfloor k/2\rfloor}\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^{k-2\lfloor k/2\rfloor} N\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^{n-k-2\lfloor (n-k)/2\rfloor}\epsilon^{2\lfloor (n-k)/2\rfloor}\|f\|_{L^1} \lesssim_{\|V\|_{\mathcal{K}},\epsilon} \epsilon^n N\|f\|_{L^1},
\end{aligned}$$

where $\lfloor a \rfloor$ is the largest integer less than or equal to a. Therefore, we conclude that

$$\|\mathcal{P}_N^n f\|_{L^\infty} \lesssim N \int_{\mathbb{R}} \varphi_N(\lambda)(n+1)\epsilon^n N \|f\|_{L^1} d\lambda \lesssim (n+1)\epsilon^n N^3 \|f\|_{L^1}.$$

□

4. Proof of Theorem 1.2 (*ii*): Low Frequencies

Now we consider low frequencies. If N is not large, the argument in the previous section does not guarantee the convergence of the formal series (2.2). This convergence issue can be solved by introducing a new formal series expansion for $\mathcal{P}_N$.

4.1. **Preliminaries.** We begin with some preliminary lemmas. For $\lambda, \lambda_0 \in \mathbb{R}$, we define the difference operator B_{λ,λ_0} by

$$B_{\lambda,\lambda_0} := VR_0^+(\lambda^2) - VR_0^+(\lambda_0^2) = V(R_0^+(\lambda^2) - R_0^+(\lambda_0^2)).$$

Lemma 4.1 (Bound for B_{λ,λ_0})**.** *Let $V \in \mathcal{K}_0$. For $\epsilon > 0$, there exist $\delta = \delta(\epsilon, V) > 0$ and an integral operator B in $\mathcal{L}(L^1)$ with kernel $B(x,y)$ such that for all $\lambda_0 \in \mathbb{R}$,*

$$|B_{\lambda,\lambda_0}(x,y)| \leq B(x,y) \text{ for } |\lambda - \lambda_0| < \delta,\ \|B\|_{\mathcal{L}(L^1)} = \|B(x,y)\|_{L_y^\infty L_x^1} < \epsilon.$$

Proof. Fix $\epsilon > 0$, and choose a bounded compactly supported function V_ϵ such that $\|V - V_\epsilon\|_{\mathcal{K}} < \epsilon$. Let $\delta > 0$ be a small number to be chosen later. Then, $V_\epsilon(R_0^+(\lambda^2) - R_0^+(\lambda_0^2))$ approximates $V(R_0^+(\lambda^2) - R_0^+(\lambda_0^2))$ with $O(\epsilon)$-error in $\mathcal{L}(L^1)$. Thus, by the mean value theorem, we have

$$\begin{aligned}
&\sup_{|\lambda-\lambda_0|<\delta} |B_{\lambda,\lambda_0}(x,y)| = \sup_{|\lambda-\lambda_0|<\delta} \Big| \frac{V(x)(e^{i\lambda|x-y|} - e^{i\lambda_0|x-y|})}{4\pi|x-y|} \Big| \\
&\leq \sup_{|\lambda-\lambda_0|<\delta} \Big| \frac{V_\epsilon(x)(e^{i\lambda|x-y|} - e^{i\lambda_0|x-y|})}{4\pi|x-y|} \Big| + \frac{|(V-V_\epsilon)(x)|}{2\pi|x-y|} \leq \frac{|V_\epsilon(x)|\delta}{4\pi} + \frac{|(V-V_\epsilon)(x)|}{2\pi|x-y|}.
\end{aligned}$$

Define the integral operator B with kernel $B(x,y) := \frac{|V_\epsilon(x)|\delta}{4\pi} + \frac{|(V-V_\epsilon)(x)|}{2\pi|x-y|}$, and take $\delta = \epsilon\|V_\epsilon\|_{L^1}^{-1}$. Then, $\|B(x,y)\|_{L_y^\infty L_x^1} < \epsilon$. □

Note that by the assumptions of this paper, $(I + VR_0^+(\lambda^2))$ is invertible in $\mathcal{L}^1(L^1)$ for all $\lambda \in \mathbb{R}$. Indeed, if it is not for some λ, then λ must be an eigenvalue or a resonance (contradiction!)[3]. Hence, for $\lambda \in \mathbb{R}$, we can define

$$S_\lambda := (I + VR_0^+(\lambda^2))^{-1};\ \tilde{S}_\lambda := (S_\lambda - I) = (I + VR_0^+(\lambda^2))^{-1} - I.$$

Lemma 4.2 (Uniform bound for $\tilde{S}_\lambda$)**.** *If $V \in \mathcal{K}_0$ and H has no eigenvalues or resonances on $[0, +\infty)$, then $\tilde{S}_\lambda$ is an integral operator with kernel $\tilde{S}_\lambda(x,y)$:*

$$\tilde{S} := \sup_{\lambda \in \mathbb{R}} \|\tilde{S}_\lambda\|_{\mathcal{L}(L_1)} = \sup_{\lambda \in \mathbb{R}} \|\tilde{S}_\lambda(x,y)\|_{L_y^\infty L_x^1} < \infty. \tag{4.1}$$

Proof. The uniform boundedness of $\tilde{S}_\lambda$ follows from [3, Theorem 2]. It remains to show that $\tilde{S}_\lambda$ is an integral operator. By algebra,

$$\tilde{S}_\lambda = (I + VR_0^+(\lambda^2))^{-1} - I = -(I + VR_0^+(\lambda^2))^{-1}VR_0^+(\lambda) = -S_\lambda VR_0^+(\lambda^2).$$

Consider $F_V(x;y,\lambda) := V(x)\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$ as a function of x with parameters $y \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, which is bounded in L_x^1, uniformly in y and λ: $\|F_V(x;y,\lambda)\|_{L_x^1} = \|\frac{V(x)}{4\pi|x-y|}\|_{L_x^1} \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}$. Hence, $s_0(x;y,\lambda) := -[S_\lambda F_V(\cdot;y,\lambda)](x)$ is also a uniformly bounded L_x^1-"function," that is,

$$\sup_{\lambda\in\mathbb{R}} \sup_{y\in\mathbb{R}^3} \|s_0(x;y,\lambda)\|_{L_x^1} < \infty.$$

Then, by the Fubini theorem and the duality,

$$\begin{aligned}
&\int_{\mathbb{R}_x^3} \Big(\int_{\mathbb{R}_y^3} s_0(x;y,\lambda) f(y) dy \Big) \overline{g(x)} dx = -\int_{\mathbb{R}_y^3} \Big(\int_{\mathbb{R}_x^3} [S_\lambda F_V(x;y,\lambda)] \overline{g(x)} dx \Big) f(y) dy \\
&= -\int_{\mathbb{R}_y^3} \Big(\int_{\mathbb{R}_x^3} F_V(x;y,\lambda) \overline{(S_\lambda^* g)(x)} dx \Big) f(y) dy = -\langle VR_0^+(\lambda) f, S_\lambda^* g\rangle_{L^2} \\
&= -\langle S_\lambda VR_0^+(\lambda) f, g\rangle_{L^2} = \langle \tilde{S}_\lambda f, g\rangle_{L^2} = \int_{\mathbb{R}_x^3} \Big(\int_{\mathbb{R}_y} \tilde{S}_\lambda(x,y) f(y) dy \Big) \overline{g(x)} dx,
\end{aligned}$$

for all $f \in L^1$ and $g \in L^\infty$. We thus conclude that $\tilde{S}_\lambda(x,y) = s_0(x;y,\lambda)$ satisfies (4.1). $\square$

4.2. **Construction of the formal series.** Let

$$\epsilon := ((\tilde{S}+1)^2\|V\|_{\mathcal{K}})^{-1}, \tag{4.2}$$

where $\tilde{S}$ is given by Lemma 4.2, and then take $\delta = \delta(\epsilon) \ll 1$ from Lemma 4.1. Replacing smaller δ if necessary, we may let δ be dyadic. We set $N_0 := \delta/2$, and consider $N \leq N_0$.

As we did in Section 2, we write

$$\begin{aligned}
\mathcal{P}_N - P_N &= \frac{1}{\pi}\int_0^\infty \chi_N(\sqrt{\lambda}) \operatorname{Im} R_V^+(\lambda) d\lambda - \frac{1}{\pi}\int_0^\infty \chi_N(\sqrt{\lambda}) \operatorname{Im} R_0^+(\lambda) d\lambda \\
&= \frac{N}{\pi}\int_{\mathbb{R}} \varphi_N(\lambda) \operatorname{Im}[R_0^+(\lambda^2)(I+VR_0^+(\lambda^2))^{-1} - R_0^+(\lambda^2)] d\lambda,
\end{aligned} \tag{4.3}$$

where P_N is the standard Littlewood-Paley projection, $\varphi_N = \varphi(\frac{\cdot}{N})$ and $\varphi(\lambda)$ is the odd extension of $\lambda\chi(\lambda)1_{[0,+\infty)}$. Given $\lambda_0 \in \mathbb{R}$, we expand the Neumann series of $(I+VR_0^+(\lambda^2))^{-1}$ about $\lambda = \lambda_0$:

$$\begin{aligned}
(I+VR_0^+(\lambda^2))^{-1} &= (I+VR_0^+(\lambda_0^2) + B_{\lambda,\lambda_0})^{-1} = [(I+B_{\lambda,\lambda_0}S_{\lambda_0})(I+VR_0^+(\lambda_0^2))]^{-1} \\
&= (I+VR_0^+(\lambda_0^2))^{-1}(I+B_{\lambda,\lambda_0}S_{\lambda_0})^{-1} \text{``}=\text{''} S_{\lambda_0}\sum_{n=0}^\infty (-B_{\lambda,\lambda_0}S_{\lambda_0})^n.
\end{aligned} \tag{4.4}$$

Plugging (4.4) with $\lambda_0 = 0$ into (4.3), we construct a formal series for $\mathcal{P}_N$:

$$\mathcal{P}_N - P_N \text{``}=\text{''} \sum_{n=0}^\infty \mathcal{P}_N^n, \tag{4.5}$$

where

$$\mathcal{P}_N^0 := \frac{N}{\pi}\int_{\mathbb{R}} \varphi_N(\lambda)\,\mathrm{Im}[R_0^+(\lambda^2)\tilde{S}_0]d\lambda$$

and

$$\mathcal{P}_N^n := (-1)^n\frac{N}{\pi}\int_{\mathbb{R}} \varphi_N(\lambda)\,\mathrm{Im}[R_0^+(\lambda^2)S_0(B_{\lambda,0}S_0)^n]d\lambda.$$

Remark 4.3. By Lemma 4.1 and 4.2, if δ are sufficiently small, then $\|B_{\lambda,0}S_0\|_{\mathcal{L}(L^1)} \ll 1$ for $|\lambda| \leq \delta/2$. Hence, the formal series (4.5) is expected to be absolutely convergent.

4.3. **Reduction to the decay estimate and the summability lemma.** One drawback of the new formal series expansion is that $\mathcal{P}_N^n(x,y)$ does not have an off-diagonal decay factor in its integral representation. For example, the integral representation of $\mathcal{P}_N^1(x,y)$ has

$$\check{\varphi}_N(|x_0-x_1|+|x_2-x_3|) - \check{\varphi}_N(|x_0-x_1|), \text{ with } x_0 = x \text{ and } x_4 = y, \tag{4.6}$$

which is analogous to $\check{\varphi}_N(\sigma_n)$ in (2.3). However, unlike $\check{\varphi}_N(\sigma_n)$, (4.6) does not decay away from $x = y$. Nevertheless, one can still hope to prove some decay estimate, since (4.6) decays rapidly away from $x = x_1$.

To enjoy the decay away from $x = x_1$, we introduce the intermediate kernel (with additional x_1 variable) such that

$$\mathcal{P}_N^n(x,y) = \int_{\mathbb{R}^3} \frac{(-1)^n N}{4\pi^2|x-x_1|}\mathcal{P}_N^n(x,x_1,y)dx_1. \tag{4.7}$$

Precisely, we define

$$\mathcal{P}_N^n(x,x_1,y) := \begin{cases} \int_{\mathbb{R}} \varphi_N(\lambda)\,\mathrm{Im}[e^{i\lambda|x-x_1|}\tilde{S}_0(x_1,y)]d\lambda & \text{for } n=0; \\ \int_{\mathbb{R}} \varphi_N(\lambda)\,\mathrm{Im}\Big[e^{i\lambda|x-x_1|}\{S_0(B_{\lambda,0}S_0)^n\}(x_1,y)\Big]d\lambda & \text{for } n\geq 1. \end{cases} \tag{4.8}$$

For Theorem 1.2 (ii), it suffices to show the following two lemmas:

Lemma 4.4 (Decay estimate)**.** *Let ϵ, N_0 and $\mathcal{P}_N^n(x,x_1,y)$ as above. There exists $K_1^n(x_1,y)$ such that for $N \leq N_0$,*

$$|\mathcal{P}_N^n(x,x_1,y)| \lesssim \frac{N\|\chi\|_{W^{m,1}}K_1^n(x_1,y)}{\langle N(x-x_1)\rangle^m}, \tag{4.9}$$

$$\|K_1^n(x_1,y)\|_{L_y^\infty L_{x_1}^1} \lesssim (\tilde{S}+1)^{n+1}\Big(\frac{\|V\|_{\mathcal{K}}}{2\pi}\Big)^n. \tag{4.10}$$

Lemma 4.5 (Summability)**.** *Let ϵ, N_0 and $\mathcal{P}_N^n(x,x_1,y)$ as above. There exists $K_2^n(x_1,y)$ such that for $0 < N \leq N_0$,*

$$|\mathcal{P}_N^n(x,x_1,y)| \lesssim NK_2^n(x_1,y), \tag{4.11}$$

$$\|K_2^n(x_1,y)\|_{L_y^\infty L_{x_1}^1} \lesssim \epsilon^n(\tilde{S}+1)^{n+1}. \tag{4.12}$$

Proof of Theorem 1.2 (ii), assuming Lemma 4.4 and 4.5. By (4.9) and (4.11), we have

$$\begin{aligned}|\mathcal{P}_N^n(x,x_1,y)| &\lesssim \Big(\frac{N\|\chi\|_{W^{2m,1}}}{\langle N(x-x_1)\rangle^{2m}}K_1^n(x_1,y)\Big)^{1/2}(NK_2^n(x_1,y))^{1/2}\\ &= \frac{N\|\chi\|_{W^{2m,1}}^{1/2}}{\langle N(x-x_1)\rangle^{m}}[K_1^n(x_1,y)K_2^n(x_1,y)]^{1/2} = \frac{N\|\chi\|_{W^{2m,1}}^{1/2}K(x_1,y)}{\langle N(x-x_1)\rangle^{m}},\end{aligned}$$

where $K(x_1,y):=\sum_{n=0}^\infty[K_1^n(x_1,y)K_2^n(x_1,y)]^{1/2}$. It remains to show $K(x_1,y)\in L_y^\infty L_{x_1}^1$ (see (4.5), (4.7) and (4.8)). Indeed, by (4.10), (4.12) and the choice of ϵ (see (4.2)), we obtain

$$\|K(x_1,y)\|_{L^1_{x_1}}\lesssim\sum_{n=0}^\infty(\tilde{S}+1)^{\frac{n+1}{2}}\Big(\frac{\|V\|_{\mathcal{K}}}{2\pi}\Big)^{\frac{n}{2}}\epsilon^{\frac{n}{2}}(\tilde{S}+1)^{\frac{n+1}{2}}=\sum_{n=0}^\infty\frac{\tilde{S}+1}{(\sqrt{2\pi})^n}<\infty.$$

□

4.4. **Proof of Lemma 4.4 and 4.5** ($n=0$)**.** Since $\varphi_N=\varphi(\frac{\cdot}{N})\in C_c^\infty$, we have

$$\begin{aligned}|\mathcal{P}_N^0(x,x_1,y)|&\le\Big|\int_{\mathbb{R}}\varphi_N(\lambda)e^{i\lambda|x-x_1|}\tilde{S}_0(x_1,y)d\lambda\Big|=\check{\varphi}_N(|x-x_1|)|\tilde{S}_0(x_1,y)|\\ &\lesssim\frac{N\|\varphi\|_{W^{m,1}}}{\langle N(x-x_1)\rangle^m}|\tilde{S}_0(x_1,y)|\lesssim\frac{N\|\chi\|_{W^{m,1}}}{\langle N(x-x_1)\rangle^m}|\tilde{S}_0(x_1,y)|.\end{aligned}$$

Define $K_1^0(x_1,y)=K_2^0(x_1,y)=\tilde{S}_0(x_1,y)$. Then $\mathcal{P}_N^0(x,x_1,y)$ satisfies (4.9) and (4.11).

4.5. **Proof of Lemma 4.4** ($n\ge1$)**.** First, splitting all $B_{\lambda,0}$ into $VR_0^+(\lambda^2)$ and $(-VR_0^+(0))$ in $\mathcal{P}_N^n(x,x_1,y)$, we write $\mathcal{P}_N^n(x,x_1,y)$ as the sum of 2^n copies of

$$\operatorname{Im}\int_{\mathbb{R}}\varphi_N(\lambda)e^{i\lambda|x-x_1|}\{S_0VR_0^+(\alpha_1\lambda^2)S_0\cdots VR_0^+(\alpha_n\lambda^2)S_0\}(x_1,y)d\lambda \tag{4.13}$$

up to $\pm$, where $\alpha_k=0$ or 1 for each $k=1,...,n$. By splitting all S_0 into I and $\tilde{S}_0$ in (4.13), we further decompose (4.13) into the sum of 2^{n+1} kernels.

Among them, let us consider the two representative terms:

$$\operatorname{Im}\int_{\mathbb{R}}\varphi_N(\lambda)e^{i\lambda|x-x_1|}\{\tilde{S}_0VR_0^+(\alpha_1\lambda^2)\tilde{S}_0\cdots VR_0^+(\alpha_n\lambda^2)\tilde{S}_0\}(x_1,y)d\lambda; \tag{4.14}$$

$$\operatorname{Im}\int_{\mathbb{R}}\varphi_N(\lambda)e^{i\lambda|x-x_1|}\{VR_0^+(\alpha_1\lambda^2)\cdots VR_0^+(\alpha_n\lambda^2)\}(x_1,y)d\lambda. \tag{4.15}$$

Because both $\tilde{S}_0$, $R_0^+(\alpha_k\lambda)$ are integral operators, by the free resolvent formula, we can write (4.14) in the integral form as

$$\begin{aligned}&\operatorname{Im}\int_{\mathbb{R}}\int_{\mathbb{R}^{6n}}\varphi_N(\lambda)\prod_{k=1}^{n+1}\tilde{S}_0(x_{2k-1},x_{2k})\prod_{k=1}^nV(x_{2k})\frac{\prod_{k=0}^n e^{i\alpha_k\lambda|x_{2k}-x_{2k+1}|}}{\prod_{k=1}^n4\pi|x_{2k}-x_{2k+1}|}d\mathbf{x}_{(2,2n+1)}d\lambda\\ &=\operatorname{Im}\int_{\mathbb{R}^{6n}}\check{\varphi}_N(\tilde{\sigma}_n)\prod_{k=1}^{n+1}\tilde{S}_0(x_{2k-1},x_{2k})\frac{\prod_{k=1}^nV(x_{2k})}{\prod_{k=1}^n4\pi|x_{2k}-x_{2k+1}|}d\mathbf{x}_{(2,2n+1)}\end{aligned}$$

where $x_0 := x$, $x_{2n+2} := y$, $\alpha_0 := 1$ and $\tilde{\sigma}_n := \sum_{k=0}^{n} \alpha_k |x_{2k} - x_{2k+1}|$. We observe that

$$|\check{\varphi}_N(\tilde{\sigma}_n)| = |N\check{\varphi}(N\tilde{\sigma}_n)| \lesssim \frac{N\|\chi\|_{W^{m,1}}}{\langle N\tilde{\sigma}_n\rangle^m} \lesssim \frac{N\|\chi\|_{W^{m,1}}}{\langle N(x_0-x_1)\rangle^m}, \tag{4.16}$$

since $\tilde{\sigma}_n \geq |x_0 - x_1|$ and $\varphi(\lambda) = \lambda\chi(\lambda)$. Applying (4.16) to (4.14), we get the arbitrary polynomial decay away from $x_0 = x_1$:

$$|(4.14)| \lesssim \frac{N\|\chi\|_{W^{m,1}}}{\langle N(x_0-x_1)\rangle^m} K^n_{(4.14)}(x_1, y)$$

where

$$\begin{aligned} K^n_{(4.14)}(x_1, y) :&= \int_{\mathbb{R}^{6n}} \frac{\prod_{k=1}^{n+1} |\tilde{S}_0(x_{2k-1}, x_{2k})| \prod_{k=1}^{n} |V(x_{2k})|}{\prod_{k=1}^{n} 4\pi |x_{2k} - x_{2k+1}|} d\mathbf{x}_{(2,2n+1)} \\ &= \{|\tilde{S}_0|(|V|(-\Delta)^{-1}|\tilde{S}_0|)^n\}(x_1, y) \end{aligned}$$

and $|\tilde{S}_0|$ is the integral operator with kernel $|\tilde{S}_0(x, y)|$. Then, since $|\tilde{S}_0|(|V|(-\Delta)^{-1}|\tilde{S}_0|)^n$ is an integral operator, by (4.1) and (3.2), it follows that

$$\|K^n_{(4.14)}(x_1, y)\|_{L^\infty_y L^1_{x_1}} = \||\tilde{S}_0|(|V|(-\Delta)^{-1}|\tilde{S}_0|)^n f\|_{L^1} \lesssim \tilde{S}^{n+1}\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^n \|f\|_{L^1}.$$

Similarly, we write (4.15) as

$$\begin{aligned} &\operatorname{Im} \int_{\mathbb{R}} \int_{\mathbb{R}^{3n-3}} \varphi_N(\lambda) \frac{\prod_{k=1}^{n} V(x_k) \prod_{k=0}^{n} e^{i\alpha_k\lambda|x_k - x_{k+1}|}}{\prod_{k=1}^{n} 4\pi|x_k - x_{k+1}|} d\mathbf{x}_{(2,n)} d\lambda \\ &= \operatorname{Im} \int_{\mathbb{R}^{3n-3}} \check{\varphi}_N(\tilde{\sigma}_n) \frac{\prod_{k=1}^{n} V(x_k)}{\prod_{k=1}^{n} 4\pi|x_k - x_{k+1}|} d\mathbf{x}_{(2,n)} \end{aligned}$$

where $x_0 := x$, $x_{n+1} := y$, $\alpha_0 := 1$ and $\tilde{\sigma}_n := \sum_{k=0}^{n} \alpha_k |x_k - x_{k+1}|$. By (4.16), we obtain

$$|(4.15)| \lesssim \frac{N\|\chi\|_{W^{m,1}}}{\langle N(x_0-x_1)\rangle^m} K^n_{(4.15)}(x_1, y)$$

where

$$K^n_{(4.15)}(x_1, y) := \int_{\mathbb{R}^{3n-3}} \frac{\prod_{k=1}^{n} |V(x_k)|}{\prod_{k=1}^{n} 4\pi|x_k - x_{k+1}|} d\mathbf{x}_{(2,n)} = (|V|(-\Delta)^{-1})^n(x_1, y)$$

By the definition of the global Kato norm, we get

$$\|K^n_{(4.15)}(x_1, y)\|_{L^\infty_y L^1_{x_1}} \leq \Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^n.$$

By the same way, we estimate other kernels, and define $K^n_1(x_1, y)$ to be the sum of all upper bounds including $K_{(4.14)}(x_1, y)$ and $K_{(4.15)}(x_1, y)$. Then it satisfies (4.9) and (4.10).

4.6. **Proof of Lemma 4.5** ($n \geq 1$)**.** We define

$$K_2^n(x_1, y) := \{(I + |\tilde{S}_0|)(B(I + |\tilde{S}_0|))^n\}(x_1, y),$$

where B is an integral operator defined in Lemma 4.1 and $|\tilde{S}_0|$ is the integral operator with $|\tilde{S}_0(x, y)|$. Then, since $|\operatorname{supp}\varphi_N| \sim N$, (4.11) follows from Lemma 4.1 and the definition (4.8). To show (4.12), splitting $(I + |\tilde{S}_0|)$ into $|\tilde{S}_0|$ and I in $K_2^n(x_1, y)$ and get 2^{n+1} terms:

$$K_2^n(x_1, y) \leq \{|\tilde{S}_0|(B|\tilde{S}_0|)^n\}(x_1, y) + \cdots + B^n(x_1, y).$$

Since $|\tilde{S}_0|$ and B are integral operators, by Lemma 4.1 and 4.2, we have, for example,

$$\begin{aligned}\|\{|\tilde{S}_0|(B|\tilde{S}_0|)^n\}(x_1, y)\|_{L_y^\infty L_{x_1}^1} &= \||\tilde{S}_0|(B|\tilde{S}_0|)^n\|_{\mathcal{L}(L^1)} \leq \tilde{S}^{n+1}\epsilon^n;\\ \|B^n(x_1, y)\|_{L_y^\infty L_{x_1}^1} &= \|B^n\|_{\mathcal{L}(L^1)} \leq \epsilon^n.\end{aligned}$$

Similarly, we can estimate remaining terms. Collecting all, we prove (4.12).

5. Proof of Theorem 1.2 (*iii*): Medium Frequencies

We only sketch the proof of Theorem 1.2 (*iii*), since it closely follows from the argument in the previous section. Let ϵ, δ, N_0 and N_1 be given in Section 3 and 4, and choose $N_0 < N < N_1$. Let $\psi \in C_c^\infty$ such that $\operatorname{supp}\psi \subset [-\delta, \delta]$, $\psi(\lambda) = 1$ for $|\lambda| \leq \frac{\delta}{3}$ and $\sum_{j=1}^\infty \psi(\cdot - \lambda_j) \equiv 1$ on $(0, +\infty)$, where $\lambda_j := j\delta$. We denote $\varphi_N^j = \varphi_N \psi(\cdot - \lambda_j)$ so that $\varphi_N = \sum_{j=N/2\delta}^{2N/\delta} \varphi_N^j$. Splitting (4.3) into

$$\mathcal{P}_N - P_N = \sum_{j=N/2\delta}^{2N/\delta} \frac{N}{\pi} \int_{\mathbb{R}} \varphi_N^j(\lambda) \operatorname{Im}[R_0^+(\lambda^2)(I + VR_0^+(\lambda^2))^{-1} - R_0^+(\lambda^2)]d\lambda,$$

and plugging (4.4) with $\lambda_0 = \lambda_j$, we obtain the formal series expansion:

$$\mathcal{P}_N - P_N \text{ “}=\text{” } \sum_{n=0}^\infty \mathcal{P}_N^n, \tag{5.1}$$

where

$$\mathcal{P}_N^0 := \sum_{j=N/2\delta}^{2N/\delta} \frac{N}{\pi} \int_{\mathbb{R}} \varphi_N^j(\lambda) \operatorname{Im}[R_0^+(\lambda^2)\tilde{S}_{\lambda_j}]d\lambda$$

and

$$\mathcal{P}_N^n := \sum_{j=N/2\delta}^{2N/\delta} (-1)^n \frac{N}{\pi} \int_{\mathbb{R}} \varphi_N^j(\lambda) \operatorname{Im}[R_0^+(\lambda^2)S_{\lambda_j}(B_{\lambda,\lambda_j}S_{\lambda_j})^n]d\lambda.$$

We also define the intermediate kernel $\mathcal{P}_N^n(x, x_1, y)$ by

$$\mathcal{P}_N^n(x, x_1, y) := \begin{cases} \displaystyle\sum_{j=N/2\delta}^{2N/\delta} \int_{\mathbb{R}} \varphi_N^j(\lambda) \operatorname{Im}[e^{i\lambda|x-x_1|}\tilde{S}_{\lambda_j}(x_1, y)]d\lambda & \text{for } n = 0;\\ \displaystyle\sum_{j=N/2\delta}^{2N/\delta} \int_{\mathbb{R}} \varphi_N^j(\lambda) \operatorname{Im}\left[e^{i\lambda|x-x_1|}\{S_{\lambda_j}(B_{\lambda,\lambda_j}S_{\lambda_j})^n\}(x_1, y)\right]d\lambda & \text{for } n \geq 1.\end{cases}$$

Then,

$$\mathcal{P}_N^n(x,y)=\int_{\mathbb{R}^3}\frac{(-1)^nN}{4\pi^2|x-x_1|}\mathcal{P}_N^n(x,x_1,y)dx_1.$$

By the argument of the previous section, for Theorem 1.2 (iii), it suffices to show the following two lemmas.

Lemma 5.1 (Decay estimate)**.** *Let ϵ, N_0, N_1 and $\mathcal{P}_N^n(x,x_1,y)$ as above. For $N_0<N<N_1$, there exists $K_{N,1}^n(x_1,y)$ such that*

$$|\mathcal{P}_N^n(x,x_1,y)|\lesssim\frac{N\|\chi\|_{W^{m,1}}K_{N,1}^n(x_1,y)}{\langle N(x-x_1)\rangle^m},\ \|K_{N,1}^n(x_1,y)\|_{L_y^\infty L_{x_1}^1}\lesssim(\tilde{S}+1)^{n+1}\Big(\frac{\|V\|_{\mathcal{K}}}{2\pi}\Big)^n.$$

Proof. Observe that

$$|\sigma^m(\varphi_N^j)^\vee(\sigma)|=|(\nabla^m\varphi_N^j)^\vee(\sigma)|\leq\|\nabla^m\varphi_N^j\|_{L^1}\lesssim\cdots\lesssim\|\chi\|_{W^{m,1}}.$$

Hence, Lemma 5.1 follows from the argument in Lemma 4.4, except that now we need to replace $\tilde{S}_0$ by $\tilde{S}_{\lambda_j}$, apply

$$|(\varphi_N^j)^\vee(\tilde{\sigma}_n)|\lesssim\frac{\|\chi\|_{W^{m,1}}}{\langle N(x-x_1)\rangle^m},$$

instead of (4.16), and sum in j. □

Lemma 5.2 (Summability)**.** *Let ϵ, N_0, N_1 and $\mathcal{P}_N^n(x,x_1,y)$ as above. For each $N_0<N<N_1$, there exists $K_{N,2}^n(x_1,y)$ such that*

$$|\mathcal{P}_N^n(x,x_1,y)|\lesssim NK_{N,2}^n(x_1,y),\ \|K_{N,2}^n(x_1,y)\|_{L_y^\infty L_{x_1}^1}\lesssim\epsilon^n(\tilde{S}+1)^{n+1}.$$

Proof. Replace $\tilde{S}_0$ by $\tilde{S}_{\lambda_j}$ in the proof of Lemma 4.5. Then Lemma 5.2 follows. □

Proof of Corollary 1.3. It suffices to show that for $f\in L^1\cap L^2$,

$$\|\mathcal{P}_Nf\|_{L^1}\lesssim\|f\|_{L^1},\ \|\mathcal{P}_Nf\|_{L^\infty}\lesssim N^3\|f\|_{L^1}.$$

Indeed, if they are true, interpolation of the two estimates and the trivial bound $\|\mathcal{P}_Nf\|_{L^2}\leq\|f\|_{L^2}$ gives Corollary 1.3. We only consider the case $N<N_0$. Other cases follows similarly. First, by Theorem 1.2 (ii), we have

$$\begin{aligned}\|\mathcal{P}_Nf\|_{L^1}&\lesssim\Big\|\int_{\mathbb{R}^6}\frac{N^2K(x_1,y)|f(y)|}{|x-x_1|\langle N(x-x_1)\rangle^3}dx_1dy\Big\|_{L_x^1}\\&\leq\int_{\mathbb{R}^6}\Big\|\frac{N^2}{|x-x_1|\langle N(x-x_1)\rangle^3}\Big\|_{L_x^1}K(x_1,y)|f(y)|dx_1dy\\&\lesssim\|K(x_1,y)\|_{L_y^\infty L_{x_1}^1}\|f\|_{L^1}\lesssim\|f\|_{L^1}.\end{aligned}$$

By the same argument and duality,

$$\|\mathcal{P}_Nf\|_{L^2}\lesssim N^{3/2}\|f\|_{L^1},\ \|\mathcal{P}_Nf\|_{L^\infty}\lesssim N^{3/2}\|f\|_{L^2}.$$

Since the proofs of Lemma 4.4 and 4.5 do not depend on the choice of χ. Indeed, if we define $\tilde{\mathcal{P}}_N$ using $\chi_* \in C_c^\infty$ such that $\chi_* = 1$ on $\operatorname{supp}\chi$ and $\operatorname{supp}\chi_* \subset (\frac{2}{5}, \frac{21}{10})$, then $\tilde{\mathcal{P}}_N$ also satisfies the above estimates. Thus,

$$\|\mathcal{P}_N f\|_{L^\infty} = \|\tilde{\mathcal{P}}_N \mathcal{P}_N f\|_{L^\infty} \lesssim N^{3/2}\|\mathcal{P}_N f\|_{L^2} \lesssim N^3\|f\|_{L^1}.$$

$\square$

6. Homogeneous Sobolev Inequality: Proof of Theorem 1.6

Since $H^{-\frac{s}{2}}\mathcal{P}_N$ has a symbol $\lambda^{-\frac{s}{2}}\tilde{\chi}_N(\lambda)$, replacing the role of χ by $\lambda^{-s}\chi$ in Theorem 1.2, one can show that there exist $N_0 \ll 1$, $N_1 \gg 1$ and $K(x_1, y) \in L_y^\infty L_{x_1}^1$ such that

$$|H^{-\frac{s}{2}}\mathcal{P}_N(x,y)| \lesssim \frac{N^{3-s}}{\langle N(x-y)\rangle^3}, \text{ for } N \geq N_1,$$

$$|H^{-\frac{s}{2}}\mathcal{P}_N(x,y) - |\nabla|^{-s}P_N(x,y)| \lesssim \int_{\mathbb{R}^3} \frac{N^{2-s}K(x_1,y)}{|x-x_1|\langle N(x-x_1)\rangle^2}dx_1, \text{ for } N \leq N_0.$$

For $x, y \in \mathbb{R}^3$, let N_* be the dyadic number such that $N_* \leq |x-y|^{-1} < 2N_*$. Observe that

$$\begin{aligned}|H^{-\frac{s}{2}}\mathcal{P}_{\geq N_1}(x,y)| &\leq \sum_{N\geq N_1}|H^{-\frac{s}{2}}\mathcal{P}_N(x,y)| \lesssim \sum_N \frac{N^{3-s}}{\langle N(x-y)\rangle^3}\\ &\lesssim \sum_{N\leq N_*} N^{3-s} + \sum_{N>N_*}\frac{N^{-s}}{|x-y|^3} \leq (N_*)^{3-s} + \frac{(N_*)^{-s}}{|x-y|^3} \lesssim \frac{1}{|x-y|^{3-s}}.\end{aligned}$$

Thus, it follows from the fractional integration inequality that $H^{-\frac{s}{2}}\mathcal{P}_{\geq N_1}$ is bounded from L^p to L^q.

Similarly, one can show that

$$|H^{-\frac{s}{2}}\mathcal{P}_{\leq N_0}(x,y) - |\nabla|^{-s}P_{\leq N_0}(x,y)| \lesssim \int_{\mathbb{R}^3}\frac{K(x_1,y)}{|x-x_1|^{3-s}}dx_1.$$

By the endpoint fractional integration inequality [14, Theorem 1, p.119], we obtain

$$\begin{aligned}\|(H^{-\frac{s}{2}}\mathcal{P}_{\leq N_0} - |\nabla|^{-s}P_{\leq N_0}f\|_{L^{\frac{3}{3-s},\infty}} &\lesssim \Big\|\int_{\mathbb{R}^6}\frac{K(x_1,y)|f(y)|}{|x-x_1|^{3-s}}dx_1dy\Big\|_{L_x^{\frac{3}{3-s},\infty}}\\ &\lesssim \Big\|\int_{\mathbb{R}^3}K(x_1,y)|f(y)|dy\Big\|_{L^1_{x_1}} \leq \|K(x_1,y)\|_{L_y^\infty L_{x_1}^1}\|f\|_{L^1} \lesssim \|f\|_{L^1}\end{aligned}$$

and

$$\||\nabla|^{-s}P_{\leq N_0}f\|_{L^{\frac{3}{3-s},\infty}} \lesssim \|f\|_{L^1}.$$

Thus, $H^{-\frac{s}{2}}\mathcal{P}_{\leq N_0}$ is bounded from L^1 to $L^{\frac{3}{3-s},\infty}$. By duality [4, Theorem 3.7.1 and 5.3.1] and the real interpolation [4, Theorem 5.3.2], we obtain that $H^{-\frac{s}{2}}\mathcal{P}_{\leq N_0} : L^p \to L^q$.

For $N_0 < N < N_1$, replacing the role of χ by $\lambda^{-s}\chi$ in Corollary 1.3, we see that $H^{-\frac{s}{2}}\mathcal{P}_N$ is bounded from L^p to L^q. Collecting all, we complete the proof.

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Brown University
E-mail address: yhhong@math.brown.edu